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SUBMANIFOLDS OF INVARIANT MANIFOLDS OF STATIONARY MOTIONS AND THEIR PROPERTIES*

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A number of results are presented, associated with investigations of invariant manifolds of stationary motions (IMSMs) of mechanical systems with first integrals /1/, which have a dimension higher than zero in phase space. The definition of such IMSMs, methods of isolating their submanifolds, and some conditions on the first integrals which ensure the existence of such IMSMs are discussed.

1. Consider a system of ordinary differential equations with smooth right-hand sides defined in some domain of R^n:

$$x_i^{\cdot} = X_i(x_1, \ldots, x_n) \tag{1.1}$$

which has a smooth autonomous first integral $V(x_1, \ldots, x_n)$. Here and elsewhere $i, j = 1, \ldots, n$.

Suppose that the partial derivatives of this integral can be represented in the form

$$\frac{\partial V}{\partial x_i} = \sum_{l=1}^{k} \varphi_l(x) a_{li}(x) \quad (k < n) \tag{1.2}$$

where the $a_{li}(x)$ $(l = 1, \ldots, k)$ are smooth functions defined on a manifold

$$\varphi_1(x) = \varphi_2(x) = \ldots = \varphi_k(x) = 0 \tag{1.3}$$

and in its neighbourhood.

Definition 1. We say that representation (1.2) is proper if the rank of the matrix $\| a_{li}(x) \|$ on the manifold (1.3) is equal to k.

Definition 2. Each first integral $V(x_1, \ldots, x_n)$ of system (1.1) which has at least one proper representation of the form (1.2) will be called an integral of parts of the system.

Thus the square of a smooth first integral $V(x) - h = 0$ is an integral of parts of the system for those values $h = h^\circ$ for which partial derivatives $\partial V/\partial x_i$ exist and do not simultaneously vanish on the manifold $V(x) = h^\circ$. This follows directly from the formulae

$$\partial (V(x) - h)^2/\partial x_i = 2(V(x) - h)\, \partial V/\partial x_i$$

and the given definitions.

It is well-known /2/ that any non-degenerate solution of the system of stationarity equations for a smooth integral

$$\partial V/\partial x_i = f_i(x_1, \ldots, x_n) = 0 \tag{1.4}$$

is a non-degenerate invariant manifold of (1.1) and, because it always serves as a solution of system (1.1), it is often called a stationary motion. (A non-degenerate solution is one on which the Jacobian of (1.4) does not vanish).

If however the Jacobian of system (1.4) vanishes on some portion of phase space, then Eqs.(1.4) are dependent.

Suppose that amongst them the first k functions in (1.4) are independent of x:

$$f_1(x_1, \ldots, x_n) = 0, \ldots, \quad f_k(x_1, \ldots, x_n) = 0 \tag{1.5}$$

and they are "irreducible", (i.e. they cannot be expressed in the form of products of functions), then system (1.5) defines an invariant manifold of (1.1) /1/. Eliminating with the help of (1.5) some of the variables x_{i_l} $(l = 1, \ldots, k)$ from (1.1), we obtain (in a corresponding chart of the manifold (1.5) /3/) differential equations for the vector field on (1.5):

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$$\dot{x}_{i_{k+1}} = X_{i_{k+1}}(x_{i_{k+1}}, \ldots, x_{i_n}), \ldots, \dot{x}_{i_n} = X_{i_n}(x_{i_{k+1}}, \ldots, x_{i_n}) \tag{1.6}$$

In other charts of manifold (1.5) the equations are similar to (1.6).

Definition 3. An invariant manifold with dimension greater than zero whose equation is a solution of system (1.4), with a vector field defined on it by the original system of differential equations, will be called a degenerate invariant manifold of stationary motions (IMSMs).

It is clear that all perturbations of the right-hand sides of Eqs.(1.1) which leave the integral $V(x)$ unchanged, also leave Eqs.(1.5) unchanged, whilst changing the vector field on the IMSM being discussed. Hence the inclusion of the vector field in the concept of an IMSM substantially "closes the problem" for degenerate IMSMs. (The number of equations in the definition of the IMSM is equal to the dimensions of the phase space).

When using the Rouse-Lyapunov theorem and its generalizations one should also have in mind the following intuitive idea.

Assertion 1. Not every submanifold of a degenerate IMSM is itself an IMSM, or even an invariant manifold.

To obtain an example of such a submanifold in the case of the IMSM (1.5), (1.6) it is sufficient to choose constant values of the coordinates $x_{i_{k+1}} = \mathring{x}_{i_{k+1}}, \ldots, x_{i_n} = \mathring{x}_{i_n}$, such that for these coordinate values some of the

$$X_{i_d}|_0 = X_{i_d}(\mathring{x}_{i_{k+1}}, \ldots, \mathring{x}_{i_n}), \quad (d = k+1, \ldots, n)$$

do not vanish in (1.6). Then at $\mathring{x}_{i_{k+1}}, \ldots, \mathring{x}_{i_n}$ we obtain from (1.4) the submanifold

$$f_1(x_{i_1}, \ldots, x_{i_k}, \mathring{x}_{i_{k+1}}, \ldots, \mathring{x}_{i_n}) = 0, \ldots, f_k(x_{i_1}, \ldots, x_{i_k}, \mathring{x}_{i_{k+1}}, \ldots, \mathring{x}_{i_n}) = 0$$

which obviously remains a solution of problem (1.4), but will not be a solution of system (1.1).

We will now consider the case of "reducible" Eqs.(1.4).

Assertion 2. Every first integral $V(x_1, \ldots, x_n)$ of parts of the system generates a degenerate IMSM (1.3) with an associated vector field.

Proof. We write the identity

$$\frac{\partial}{\partial x_j}\left(\frac{dV}{dt}\right) \equiv 0$$

in x using the commutativity of partial differentiation operators on $V(x)$ in the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial V}{\partial x_j}\right) X_i + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{\partial X_i}{\partial x_j} \equiv 0$$

After using representation (1.2) for the derivatives of the integral of parts of the system $V(x)$ this expression enables us to write n equalities

$$\sum_{l=1}^{k} \varkappa_l a_{lj} = -\sum_{l=1}^{k} \varphi_l \left[\sum_{i=1}^{n}\left(\frac{\partial a_{lj}}{\partial x_i} X_i + a_{li}\frac{\partial X_i}{\partial x_j}\right)\right], \quad \varkappa_l = \sum_{i=1}^{n} \frac{\partial \varphi_l}{\partial x_i} X_i$$

The last equation reduces on manifold (1.3) to a system of homogeneous equations in k unknowns $\varkappa_l$ $(l = 1, \ldots, k)$. Because representation (1.2) is proper, the rank of the matrix of the system under consideration is k, and on manifold (1.3) it only has a trivial solution. This enables us to conclude that manifold (1.3) is invariant for system (1.1).

In order to obtain the corresponding IMSM (1.3), it is sufficient to determine the vector field on the given manifold. To do this it is necessary to use (1.3) to eliminate k variables x_{i_j} from Eq.(1.1). As a result one obtains differential equations of the form (1.6) which determine the required vector field on the corresponding chart. Thus, by inspecting all the charts, one can determine the vector field throughout the whole manifold.

Consider the total derivative of the integral $V(x)$ with respect to system (1.1):

$$\partial V/dt \triangleq \varphi_1(x)(a_{11}X_1 + \ldots + a_{1n}X_n) + \ldots + \varphi_k(x)(a_{k1}X_1 + \ldots + a_{kn}X_n) \tag{1.7}$$

and introduce new variables with the help of the following differential relations:

$$\varphi_l^{\cdot} = a_{l1}x_1^{\cdot} + \ldots + a_{ln}x_n^{\cdot} \quad (l = 1, \ldots, k) \tag{1.8}$$

If this last system of equations (distribution /3/) is integrable, i.e. if

$$\partial\varphi_1/\partial x_1 = a_{11}, \ldots, \partial\varphi_1/\partial x_n = a_{1n}, \ldots, \quad \partial\varphi_k/\partial x_1 = a_{k1}, \ldots, \partial\varphi_k/\partial x_n = a_{kn}$$

then in the new variables $\varphi_1, \ldots, \varphi_k, x_{k+1}, \ldots, x_n$ expression (1.7) can be written in the form

$$dV/dt = \varphi_1\varphi_1^{\cdot} + \ldots + \varphi_k\varphi_k^{\cdot}$$

Integrating the last equation we obtain

$$V = {}^1/_2\,(\varphi_1^2 + \varphi_2^2 + \ldots + \varphi_k^2) \tag{1.9}$$

Thus, in the case of an integrable distribution (1.8) variables exist in which the IMSM (1.3), (1.6) under consideration turns into a linear subspace

$$\varphi_1 = \varphi_2 = \ldots = \varphi_k = 0 \tag{1.10}$$

with associated vector field (1.6), whilst the quadratic integral of parts of system (1.9) serves as its generator.

Obviously, manifold (1.10) can be considered to be basic, and then the original IMSM (1.3) is, generally speaking, a ramified covering for it.

Integral (1.9) is sign-definite for $\varphi_1, \ldots, \varphi_k$ in a neighbourhood of (1.10), which means that on the basis of the theorem corresponding to Lyapunov's second method one can deduce the stability of the IMSM (1.10), and consequently, that of the IMSM (1.3), (1.6) as well. This proves the following theorem.

Theorem 1. If system (1.1) has a first integral of parts of the system with representation (1.2) and distribution (1.8) is integrable, then IMSM (1.3), (1.6) is Lyapunov stable.

2. We shall give a specific example of an investigation of a somewhat general mechanical system for which there exist non-quadratic first integrals of parts of the system of differential equations of motion.

Consider a system of coupled solids with a support (one of the bodies having a single fixed point).

The Lagrange function for such a system can be written as follows /4/:

$$2L = \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} J_{\alpha\beta}(q)\,\omega_\alpha\omega_\beta + 2\sum_{\alpha=1}^{3}\sum_{k=1}^{n} e_{k\alpha}(q)\,\omega_\alpha q_k^{\cdot} + \sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk}(q)\,q_j^{\cdot}q_k^{\cdot} - 2U(q_1, \ldots, q_n)$$

Here ω_α $(\alpha = 1, 2, 3)$ are the projections of the angular velocity of the supporting body onto coordinate axes tied to it, q_k $(k = 1, \ldots, n)$ are generalized coordinates giving the positions of the supported bodies relative to each other and to the supporting body and $U(q_1, \ldots, q_n)$ is the force function defining the interactions of the supported bodies. (L can also be interpreted differently, see e.g. /5/). The differential equations of motion of this system

$$\sum_{j=1}^{n} c_{ij}q_j^{\cdot\cdot} + \sum_{\alpha=1}^{3} e_{i\alpha}\omega_\alpha^{\cdot} - \sum_{k=1}^{n}\sum_{\alpha=1}^{3}\left(\frac{\partial e_{k\alpha}}{\partial q_i} - \frac{\partial e_{i\alpha}}{\partial q_k}\right) q_k^{\cdot}\omega_\alpha + \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial c_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial c_{kj}}{\partial q_i}\right) q_k^{\cdot}q_j^{\cdot} - \frac{1}{2}\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\frac{\partial J_{\alpha\beta}}{\partial q_i}\omega_\alpha\omega_\beta - \frac{\partial U}{\partial q_i} = 0 \tag{2.1}$$

$$\sum_{\alpha=1}^{3} J_{1\alpha}\omega_\alpha^{\cdot} + \sum_{k=1}^{n} e_{k1}q_k^{\cdot\cdot} + \sum_{k=1}^{n}\left(e_{k3}\omega_2 - e_{k2}\omega_3 + \sum_{\alpha=1}^{3}\frac{\partial J_{1\alpha}}{\partial q_k}\omega_\alpha\right) q_k^{\cdot} + \sum_{\alpha=1}^{3}(J_{3\alpha}\omega_2 - J_{2\alpha}\omega_3)\,\omega_\alpha + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial e_{k1}}{\partial q_j} q_k^{\cdot}q_j^{\cdot} = 0$$

$$\gamma_1^{\cdot} = \gamma_2\omega_3 - \gamma_3\omega_2 \quad (1,\ 2,\ 3)$$

(where equations that are obtained by cyclic permutations of the indices 1, 2, 3 have not been written out) have four first integrals

$$2H = \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} J_{\alpha\beta}(q)\,\omega_\alpha\omega_\beta + 2\sum_{k=1}^{n}\sum_{\alpha=1}^{3} e_{k\alpha}(q)\,q_k^{\cdot}\omega_\alpha + \tag{2.2}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk}(q) q_j^{\cdot} q_k^{\cdot} - 2U(q) = 2h, \quad V_1 = \sum_{\beta=1}^{3} \Omega_\beta \gamma_\beta = m$$

$$V_2 = \sum_{\beta=1}^{3} \Omega_\beta^2 = n^2, \quad V_3 = \sum_{\beta=1}^{3} \gamma_\beta^2 = 1,$$

$$\Omega_\beta = \sum_{\alpha=1}^{3} J_{\alpha\beta}\omega_\alpha + \sum_{k=1}^{n} e_{k\beta} q_k^{\cdot} \quad (\beta = 1, 2, 3)$$

The last three of these, as can be easily verified, are integrals of parts of the system (2.1).

We shall isolate the degenerate invariant manifolds of stationary motions which correspond to this triplet of integrals. To do this we form the combination

$$K_{(123)} = {}^1/_2 V_2 - \lambda_1 V_1 - {}^1/_2 \lambda_3 V_3$$

and write the stationarity conditions of one of the integrals of the combination in terms of the others, following Lagrange's method

$$\frac{\partial K_{(123)}}{\partial \omega_\gamma} = \sum_{\beta=1}^{3} (\Omega_\beta - \lambda_1 \gamma_\beta) J_{\beta\gamma} = 0, \quad \frac{\partial K_{(123)}}{\partial \gamma_\gamma} = -\lambda_1 \Omega_\gamma - \lambda_3 \gamma_\gamma = 0 \quad (\gamma = 1, 2, 3)$$

$$\frac{\partial K_{(123)}}{\partial q_i^{\cdot}} = \sum_{\beta=1}^{3} (\Omega_\beta - \lambda_1 \gamma_\beta) e_{i\beta} = 0$$

$$\frac{\partial K_{(123)}}{\partial q_i} = \sum_{\beta=1}^{3} (\Omega_\beta - \lambda_1 \gamma_\beta) \Big(\sum_{\alpha=1}^{3} \frac{\partial J_{\beta\alpha}}{\partial q_i} \omega_\alpha + \sum_{k=1}^{n} \frac{\partial e_{k\beta}}{\partial q_i} q_k^{\cdot} \Big) = 0$$

$$V_1 = \sum_{\beta=1}^{3} \Omega_\beta \gamma_\beta = m, \quad V_3 = \sum_{\beta=1}^{3} \gamma_\beta^2 = 1$$

with $\lambda_3 = -\lambda_1^2$ and introducing the notation

$$\varphi_1 = \Omega_1 - \lambda_1 \gamma_1, \quad \varphi_2 = \Omega_2 - \lambda_1 \gamma_2, \quad \varphi_3 = \Omega_3 - \lambda_1 \gamma_3$$

one can rewrite the stationarity conditions in the form of (1.2)

$$\frac{\partial K_{(123)}}{\partial \omega_\gamma} = \sum_{\beta=1}^{3} \varphi_\beta J_{\beta\gamma} = 0, \quad \frac{\partial K_{(123)}}{\partial \gamma_\gamma} = -\lambda_1 \varphi_\gamma = 0$$

$$\frac{\partial K_{(123)}}{\partial q_i^{\cdot}} = \sum_{\beta=1}^{3} \varphi_\beta e_{i\beta} = 0, \quad \frac{\partial K_{(123)}}{\partial q_i} = \sum_{\beta=1}^{3} \varphi_\beta \Big(\sum_{\alpha=1}^{3} \frac{\partial J_{\alpha\beta}}{\partial q_i} \omega_\alpha +$$

$$\sum_{k=1}^{n} \frac{\partial e_{k\beta}}{\partial q_i} q_k^{\cdot} \Big) = 0, \quad V_1 = \sum_{\beta=1}^{3} \Omega_\beta \gamma_\beta = m, \quad V_3 = \sum_{\beta=1}^{3} \gamma_\beta^2 = 1$$

This representation is proper, because the determinant of the matrix

$$\det \begin{Vmatrix} J_{11}(q) & J_{12}(q) & J_{13}(q) \\ J_{12}(q) & J_{22}(q) & J_{23}(q) \\ J_{13}(q) & J_{23}(q) & J_{33}(q) \end{Vmatrix}$$

is non-zero for all values of $q_k, (k = 1, \ldots, n)$ because of its mechanical meaning.

From this it follows, consistent with Assertion 2, that the equations

$$\varphi_\beta = \Omega_\beta - \lambda_1 \gamma_\beta = 0 \quad (\beta = 1, 2, 3) \tag{2.3}$$

define a family of invariant manifolds of systems (2.1). Eliminating γ_1, γ_2 and γ_3 with the aid of (2.3) from the integrals $V_1 = m$ and $V_3 = 1$, we have $m = \lambda_1 = n$.

Thus on each level surface of the integral $V_1 = m$ there lies exactly one representative of the single-parameter family (2.3).

Eliminating γ_1, γ_2 and γ_3 with the help of the same conditions from the last three differential equations of (2.1), we obtain relations coinciding with the first three equations of (2.1).

Consequently, the vector field on the invariant manifold (2.3) will be governed by Eqs. (2.1) if one drops the last three (Poisson) equations. We remark that for all values of the family's parameter $\lambda_1 = m$ the vector field will be the same. (The parameter does not occur in the differential equations).

Direct verification shows that in the case under consideration system (1.8) takes the form

$$\varphi_\beta^{\cdot} = \sum_{\alpha=1}^{3} J_{\alpha\beta}\omega_\alpha^{\cdot} + \sum_{k=1}^{n} e_{k\beta} q_k^{\cdot\cdot} - \lambda_1 \gamma_\beta^{\cdot} +$$
$$\sum_{j=1}^{n}\Big(\sum_{\alpha=1}^{3}\frac{\partial J_{\alpha\beta}}{\partial q_j}\omega_\alpha + \sum_{k=1}^{n}\frac{\partial e_{k\beta}}{\partial q_j} q_k^{\cdot}\Big) q_j^{\cdot} \quad (\beta = 1, 2, 3)$$

and is integrable.

It is possible to change to new variables $\varphi_1, \varphi_2, \varphi_3, \omega_1, \omega_2, \omega_3$ and $q_j^{\cdot} q_j$; in these variables the IMSM (2.3) takes the form of the linear subspace

$$\varphi_1 = \varphi_2 = \varphi_3 = 0 \tag{2.4}$$

with a vector field described by Eqs.(2.1) (without the three Poisson equations).

The generator of the integral $K_{(123)}$ in a neighbourhood of (2.4) becomes quadratic: $K_{(123)} = \varphi_1^2 + \varphi_2^2 + \varphi_3^2$ and on the basis of Lyapunov's second method /6/ enables one to conclude that the IMSM obtained is stable.

The stability of the IMSM (2.4) here implies the stability of the IMSM (2.3) for any $\lambda_1 \neq 0$.

3. Possessing a regular algorithm for obtaining degenerate IMSMs, one can pose the problem of finding submanifolds of stationary motions of these degenerate IMSMs.

Here two possibilities arise, corresponding to the two groups of Eqs.(1.5) and (1.6) occurring in the definition of an IMSM.

The first of these is associated with finding an IMSM of higher level for Eqs.(1.6) with the help of the first integrals belonging to these equations and lifting these IMSMs in phase space /7/.

The second way is associated with Eqs.(1.5) and consists of finding sections of the original degenerate invariant manifolds of stationary motions by level hypersurfaces of first integrals of the problem.

In both cases one has to prove that this results in finding invariant submanifolds that are indeed invariant submanifolds of stationary motions, i.e. supply stationary values for one of the first integrals or a combination of them.

In certain cases this turns out not to be the case. Not every second-level IMSM /1/ can be lifted in phase space, and even a lifted second-level IMSM may turn out not to be an IMSM there.

A generating integral for submanifolds of an IMSM obtained by the second method is the generating integral of the original IMSM completed by taking the square of that integral, which "cuts out" the required submanifold from the IMSM under consideration.

This construction gives a good interpretation of the combination of integrals, which as well as containing a linear combination of integrals contains squares of the latter, in terms of the Rouse-Lyapunov theorem.

4. To illustrate the features of isolating submanifolds of IMSMs we will give a specific investigation of such a problem for the motion of a Kovalevskaya top. Here the differential equations have the form /8/

$$2p^{\cdot} = qr,\quad 2q^{\cdot} = -pr + x_0\gamma_3,\quad r^{\cdot} = -x_0\gamma_2,\quad \gamma_1^{\cdot} = \gamma_2 r - \gamma_3 q \quad (1,\ 2,\ 3) \tag{4.1}$$

We consider the combination of the energy integral and the Kovalevskaya integral (4.1)

$$2K_{(02)} = 2p^2 + 2q^2 + r^2 + 2x_0\gamma_1 - \lambda_2\,[(p^2 - q^2 - x_0\gamma_1)^2 + (2pq - x_0\gamma_2)^2] \tag{4.2}$$

and write the stationarity condition for $K_{(02)}$ in the form

$$\partial K_{(02)}/\partial p = 2\,[1 - \lambda_2\,(p^2 + q^2 - x_0\gamma_1)]\,p + 2\lambda_2 x_0 q\gamma_2 = 0 \tag{4.3}$$
$$\partial K_{(02)}/\partial q = 2\lambda_2 pqp + 2px_0\gamma_2 + 2q\,[1 - \lambda_2\,(q^2 + x_0\gamma_1)] = 0$$
$$\partial K_{(02)}/\partial \gamma_1 = \lambda_2 p x_0 p + x_0\,[1 - \lambda_2\,(q^2 + x_0\gamma_1)] = 0$$
$$\partial K_{(02)}/\partial \gamma_2 = 2\lambda_2 q x_0 p - \lambda_2 x_0^2\gamma_2 = 0,\quad \partial K_{(02)}/\partial r = r = 0$$
$$2H = 2p^2 + 2q^2 + r^2 + 2x_0\gamma_1 = 2h$$

For $\lambda_2 \neq 0$ representation (4.3) is proper, and so defines a family of invariant manifolds of stationary motions:

$$p = r = \gamma_2 = 0,\quad 1 - \lambda_2\,(q^2 + x_0\gamma_1) = 0,\quad 2h = 1/\lambda_2 \tag{4.4}$$

The vector field on the given manifold is given by the standard method from (4.1):

$$2q^{\cdot} = x_0\gamma_3, \quad \gamma_3^{\cdot} = q\,(1 - \lambda_2 q^2)/x_0\lambda_2 \tag{4.5}$$

The family parameter λ_2 is contained both in (4.4) and (4.5). The distribution corresponding to representation (4.3) is not integrable. One can look on this family of degenerate IMSMs as on a submanifold of an invariant manifold of pendulum oscillations of a body about a horizontal principal axis y, which is described by the equations

$$r = p = \gamma_2 = 0 \tag{4.6}$$

$$\gamma_1^{\cdot} = -\gamma_3 q, \quad 2q^{\cdot} = x_0\gamma_3, \quad \gamma_3^{\cdot} = \gamma_1 q \tag{4.7}$$

obtained from (4.4) and (4.5) by a section of the latter by a level hypersurface of the energy integral.

Such IMSMs, of course, can appear in cases when one of the integrals occurring in the generating combination becomes equal to the square of another integral in that combination on the given IMSM. (In the case under consideration on the manifold $p = r = \gamma_2 = 0$ the Kovalevskaya integral is equal to the square of the energy integral).

We shall show that the invariant manifold (4.6), (4.7) is itself an IMSM, and its generating integral is obtained from (4.2) by the addition of the square of the energy integral. The combination

$$K_{(020)} = H - {}^1\!/_2\lambda_2 V_2 - {}^1\!/_2\mu\,(H - h)^2$$

gives us five stationary conditions, in which we put $\mu = -\lambda_2$ and $h = -1/\mu$ and obtain the following representation of the partial derivatives of $K_{(020)}$:

$$\partial K_{(020)}/\partial p = 4\lambda_2 x_0\gamma_1 p + 2\lambda_2 x_0 q\gamma_2 + 2\lambda_2 prr = 0$$

$$\partial K_{(020)}/\partial q = 2px_0\gamma_2 + \lambda_2 qrr = 0$$

$$\partial K_{(020)}/\partial r = [1 + \lambda_2 (H - h)]\, r = 0, \quad \partial K_{(020)}/\partial\gamma_1 = {}^1\!/_2\lambda_2 x_0 rr = 0$$

$$\partial K_{(020)}/\partial\gamma_2 = 2\lambda_2 x_0 qp - \lambda_2 x_0^2\gamma_2 = 0$$

It is clear that for $\lambda_2 \neq 0$ and $h \neq 0$ this representation is proper and defines the required invariant manifold of pendulum oscillations (4.6) and (4.7).

In order to finish with an illustration of possible anomalies, we consider the Delone IMSM defined by the equations /1/

$$p^2 - q^2 - x_0\gamma_1 = 0, \quad 2pq - x_0\gamma_2 = 0 \tag{4.8}$$

$$2p^{\cdot} = pq, \quad 2q^{\cdot} = -rp + x_0\gamma_3, \quad r^{\cdot} = -2pq, \quad \gamma_3^{\cdot} = q\,(p^2 + q^2)/x_0 \tag{4.9}$$

and find the second-level IMSM generated on the given manifold (4.8), (4.9) by the energy integral $4p^2 + r^2 = 2h$ of system (4.9). The relations

$$p = q = 0, \quad 2q^{\cdot} = x_0\gamma_3, \quad \gamma_3^{\cdot} = -q^3/x_0 \tag{4.10}$$

are of course such an invariant second-level manifold.

Lifting this IMSM with the help of formulae (4.8), (4.9) into the phase space of the original problem, we obtain

$$p = r = \gamma_2 = 0, \quad q^2 + x_0\gamma_1 = 0, \quad 2q^{\cdot} = x_0\gamma_3, \quad \gamma_3^{\cdot} = -q^3/x_0 \tag{4.11}$$

This invariant manifold of system (4.1) consists of those pendulum oscillations of the body about the horizontal y axis for which the mechanical energy $q^2 + x_0\gamma_1 = h = 0$. It is "limiting" for the family of IMSMs (4.4), (4.5) and is interesting in that, being the lifting of a second-level IMSM into the phase space, is not an IMSM there, (not having a generating integral).

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ON AVERAGING IN SYSTEMS WITH A VARIABLE NUMBER OF DEGREES OF FREEDOM*

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An averaging procedure is established for systems with a variable number of degrees of freedom which arise when considering vibrocollisional oscillations with zero velocity restitution coefficient. Compared with the method of staged integration /1, 2/ the approach presented, associated with non-analytical changes in the variables /3, 4/, widens the class of systems that can be considered. Unlike the classical averaging method /5-7/ there is a reduction in the degenerate degrees of freedom because of the specific degeneracy of the problem.

1. Consider a system described over certain times by differential relations, and at other times interval by differential and finite relations of the following form:

$$x^{\cdot} = \mu X(x, yM, t, \mu), \quad y^{\cdot}M = \mu Y(x, yM, t, \mu) M$$
$$y(2\pi n) = G(x(2\pi n), \mu), \quad x(0) = x_0 \tag{1.1}$$

Here $M = M(t)$ is a 2π-periodic piecewise-constant function (see Fig.1); here and throughout $n = 0, 1, 2, \ldots$.

We take X and Y to be bounded 2π-periodic finite-dimensional vector functions satisfying Lipschitz conditions on their first and second arguments, G is a bounded vector function with bounded partial derivatives with respect to the first argument and μ is a small parameter.

The vector function $y(t)$ is a solution of an infinite sequence of systems of differential equations, each of which acts in the time interval $2\pi n \leqslant t < 2\pi(n+1)$, after which new initial conditions are imposed.

Together with system (1.1) we consider the averaged equations

$$\xi^{\cdot} = \mu \Xi(\xi, \eta, \mu) \equiv \mu \langle X(\xi, \eta M, t, \mu)\rangle$$
$$\eta = G(\xi, \mu), \quad \xi(0) = x_0 \tag{1.2}$$

Under the given conditions we formulate the following theorem.

Theorem. If the solution of system (1.2) is given in a time interval of the order of $1/\mu$, then

$$\| x - \xi \| \leqslant C_1\mu, \quad \| yM - \eta M \| \leqslant C_2\mu \tag{1.3}$$

during that time interval, and the constants C_1 and C_2 remain bounded as $\mu \to 0$.

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